Communication for maths

On the formal presentation of algebra

Reminder of the aim of this course

- At its most general, mathematical communication is about communicating mathematical ideas, and doing so mathematically.

- To communicate these ideas requires a clarity and precision of exposition/presentation:
  - "What steps do I present in my work?"
    - "I present this step because… I don't present this step because …"

Reminder of the aim of this course

- To communicate these ideas requires a clarity and precision of exposition/presentation:
  - "How do I present a particular step?"
    - "Do I use the factor theorem or do I use the rational root test?"
    - "Do I factorise by inspection or do I use the quadratic formula?"

etc.

Reminder of the aim of this course

- Beyond this there is the style of presentation:

Style 1

Definition

Let $\mathbb{F}$ be a number system. A *polynomial in x over* $\mathbb{F}$ is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{F}$ are the *coefficients* of $f(x)$. A set of all polynomial in x over $\mathbb{F}$ is denoted $\mathbb{F}[x]$.

Reminder of the aim of this course

- Beyond this there is the style of presentation:

<u>*Style 2*</u>

The general form of a polynomial is written as

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers.

We will not study style in CIM. The above examples are just to show you what is possible in communicating maths.

Reminder of the aim of this course

- The aim of the communication for maths course is not just to serve the UPCSE maths course.
- It is not just about learning how to present maths for coursework and exams.
- The aim of this course is also more broadly to teach you some basics about the presentation of mathematics for your future academic careers.

Reminder of the aim of this course

- Some teachers may say “This is not necessary in your presentation of maths”.
- You might say, “But Chris told me to do this.”
- No. If you have shown the minimum presentation necessary, then it is up to *you* to decide what else you wish to present.

Reminder of the aim of this course

- Too little presentation makes the reader put more effort and guess work into understanding what you are saying.

- Too much presentation slows the reader down with trivial or straightforward things, and distract from the main mathematical idea and process you wish to present, and which the reader wants to read.

Reminder of the aim of this course

- The right balance of presentation makes your mathematical idea straightforward to read, learn from and understand, and a pleasure to read.
- This is not easy to do at the professional level, and many professionals do not do it well.
- I myself do not find writing-for-readability a straightforward activity.

Reminder of the aim of this course

- Ultimately it is for you
  - to learn to understand how to present and communicate maths,
  - to decide how you will present maths not only in order to communicate mathematics but also in order to make your communication readable.

How to start a mathematical sentence

- **Start each sentence with a word not a symbol**

<u>Example</u>

$f(x)$ is a function with range in $\mathbb{R}$ — **No**

A function $f(x)$ has range in $\mathbb{R}$ — **Yes**

Let $f(x)$ be a function with range in $\mathbb{R}$ — **Yes**

How to start a mathematical sentence.

Start each sentence with a word not a symbol.

Example

– "$x^2 - 6x + 8 = 0$ has two distinct roots". **No**

How to start a mathematical sentence.

Start each sentence with a word not a symbol.

Example

- "$x^2 - 6x + 8 = 0$ has two distinct roots". **No**

- "The equation $x^2 - 6x + 8 = 0$ has two distinct roots". **Yes**

or

- "The two roots of $x^2 - 6x + 8 = 0$ are distinct". **Yes**

Continuing a mathematical sentence.

Where relevant separate variables by words.

Example

- "Except for a, b is the only root of $(x - a)(x - b) = 0$".

No

Continuing a mathematical sentence.

Where relevant separate variables by words.

Example

- "Except for a, b is the only roots of $(x - a)(x - b) = 0$". **No**

- "Except for a the value b is the only roots of $(x - a)(x - b) = 0$". **Yes**

or

- "Given that a is a root of $(x - a)(x - b) = 0$ the value b is another root". **Yes**

Write complete mathematical sentences or phrases in a solution.

Example:

There is at least one error in presentation here. Where?

Find the values of x which satisfy

$$f(x) = 2x^2 + 5x - 3 = 0$$

Solution

$f(\frac{1}{2})$ and $f(-3) = 0$

$(2x-1)$ and $(x+3)$ are

factors of $f(x)$.

Hence $(2x-1)(x+3) = 0$

$\therefore (2x-1) = 0 \Rightarrow x = \frac{1}{2}$,

and $(x+3) = 0 \Rightarrow x = -3$.

Write complete mathematical sentences or phrases in a solution.

Example:

Find The values of x which satisfy

$$f(x) = 2x^2 + 5x - 3 = 0$$

<u>Solution</u>

Since $f(\frac{1}{2}) = 0$ and $f(-3) = 0$, we have $(2x-1)$ and $(x+3)$ are factors of $f(x)$.

Sentence layout

Hence $(2x-1)(x+3) = 0$

$\therefore$ $(2x-1) = 0 \Rightarrow x = \frac{1}{2}$,

and/or $(x+3) = 0 \Rightarrow x = -3$.

Write complete mathematical sentences or phrases in a solution.

Similarly,

Since $(x-2)(x-3)=0$

it follows that $x = 2$ or 3.

No

Since $(x-2)(x-3)=0$

it follows That $x=2$ or $x=3$.

Yes

Write complete mathematical sentences or phrases in a solution.

Similarly,

- “If a, b = 0 then …”. **No**

Does this mean

- “If a, then b = 0.” **?**

or

- “If a = 0 then b = 0.” **?**

or

- “If a = 0 and b = 0 then …” **?**

Write complete mathematical sentences or phrases in a solution.

Another incorrect form of presentation:

$$\text{Let } g(x) = 1 - x \;\text{ for } x < 0, = 1 + x \;\text{ for } x \geq 0$$

What is wrong with the presentation of this step, and how would you correct it?

Write complete mathematical sentences or phrases in a solution.

Correct form of presentation:

1) Let $g(x) = 1 - x$ for $x < 0$ and $g(x) = 1 + x$ for $x \geq 0$.

or

2) Let

$$g(x) = \begin{cases} 1 - x & \text{for } x < 0 \\ 1 + x & \text{for } x \geq 0 \end{cases}$$

Align your equal signs

Example

Expand $f(x) = x(x-1)(x+2)$

Solution

$$f(x) = x(x-1)(x+2)$$
$$=(x^2-x)(x+2)$$
$$= x^3+x^2-2x$$

Solution not correctly presented

Align your equal signs

Example

Expand $f(x) = x(x-1)(x+2)$

Solution

Equal signs properly aligned

$$\begin{aligned} f(x) &= x(x-1)(x+2) \\ &= (x^2-x)(x+2) \\ &= x^3+x^2-2x \end{aligned}$$

No free-standing expressions

Example: What is wrong with this solution?

Expand $f(x) = x(x-1)(x+2)$

Solution

$$x(x-1)(x+2)$$

$$(x^2-x)(x+2)$$

$$x^3+x^2-2x$$

No free-standing expressions

- Every single step of a solution is an individual mathematical sentence.
- In mathematics, individual steps/sentences have (amongst other things) the symbol "=".
- We shouldn't write isolate expressions which show no logical connection to anything.
- Use the "=" sign to make logical connections, and to form complete sentences.

No free-standing expressions

- **Example**

$$3x^2 + x - 2$$ ← Meaningless. Why?

vs $$f(x) = 3x^2 + x - 2$$ ← Meaningful.

What does it mean?

vs $$3x^2 + x - 2 = 0$$ ← Meaningful.

What does it mean?

all mean different things.

Give complete answers at each intermediate stage

Example

Using $\sqrt{x} = y$, where $y \geq 0$, find the Real Roots of

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0.$$

Solution

Let $\sqrt{x} = y$. Then $x = y^2$.

$\therefore \quad y^2 + 3y - \tfrac{1}{2} = 0$

and $\therefore \quad 2y^2 + 6y - 1 = 0$

By the quadratic formula

$$y = \frac{-6 \pm \sqrt{36-8}}{4}$$

$$= -\frac{3}{2} + \frac{\sqrt{11}}{2}$$

Not the correct answer for this step →

Give complete answer at each intermediate stage

Example

Using $\sqrt{x} = y$, where $y \geq 0$, find the Real Roots of

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and $\therefore \quad 2y^2 + 6y - 1 = 0$

By the quadratic formula

$$y = \frac{-6 \pm \sqrt{36-8}}{4}$$

$$= -\frac{3}{2} + \frac{\sqrt{11}}{2} \quad \text{and} \quad -\frac{3}{2} - \frac{\sqrt{11}}{2}$$

But $y \geq 0$ so $y = -\frac{3}{2} + \frac{\sqrt{11}}{2}$ . . . . .

Correct answer for this stage.

Justification →

Use the method stated, give the answer required

Example: "By using the rational root test find one factor of $f(x) = 2x^2 + 5x - 3 = 0$"

Solution

$f(\frac{1}{2}) = 0$, $\therefore$ $(2x-1)$ is a factor of $f(x)$.

What method has been used here?

Solution

By the rational root test $p \mid a_0$ and $q \mid a_n$.

So $p = \pm\{1, 2\}$ and $q = \pm\{1, 3\}$.

A possible root is $x = -\frac{1}{2}$. Testing $f(-\frac{1}{2})$ we have

$$f(-\tfrac{1}{2}) = \frac{2}{4} - \frac{5}{2} - 3 \neq 0$$

So $x = -\frac{1}{2}$ is NOT a root of $f(x)$.

Another possible root is $x = \frac{1}{2}$. Testing $f(\frac{1}{2})$ we obtain

$$f(\tfrac{1}{2}) = \frac{2}{4} + \frac{5}{2} - 3 = 0$$

So $x = \frac{1}{2}$ is a root of $f(x)$.

Hence $(2x - 1)$ is a factor of $f(x)$

Correct solution since it uses the method required

Justifying steps (again)

Examples

How could the following presentation of maths be improved?

$$2x^2 + \lambda x + 2 = 0$$

$$\therefore \quad \lambda^2 - 16 = 0 \quad \Rightarrow \quad \lambda = \pm 4.$$

Justifying steps (again)

Examples

Firstly, by including the missing justification as to why we are doing $\lambda^2 - 16 = 0$:

$$2x^2 + \lambda x + 2 = 0$$

For Real & equal Roots, $\Delta = 0$

$$\therefore \quad \lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4.$$

Justifying steps (again)

Examples

Secondly, by … ?

$$2x^2 + \lambda x + 2 = 0$$

For Real & equal Roots, $\Delta = 0$

$$\therefore \quad \lambda^2 - 16 = 0 \quad \Rightarrow \quad \lambda = \pm 4.$$

Justifying steps (again)

Examples

How could the following presentation of maths be improved?

$$f(x) = 2x^3 + 3x^2 - 2x - 3$$

$$\therefore f(x) = (x-1)(2x^2+5x+3)$$

Justifying steps (again)

Examples

By justifying how we got $(x-1)$:

$$f(x) = 2x^3 + 3x^2 - 2x - 3$$

Since $f(1) = 0$, $(x-1)$ is a factor of $f(x)$

$$\therefore f(x) = (x-1)(2x^2 + 5x + 3)$$

Justifying steps (again)

Examples

What is the difference between showing

$$(2x+1)(x-2) = 2x^2 - 3x - 2 \quad ①$$

and showing

$$6x^3 + 11x^2 - 3x - 2 = (2x-1)(x+2)(3x+1) \quad ②$$

?

Justifying steps (again)

Examples

① is justified automatically by the fact that expansion is defined this way, without any theorem proving such expansion.

② is only justified by the fact that there is a theorem which allows us to write in factored form: the fact theorem.

Justification steps demonstrate your mathematical understanding of why the next step is what it is.

Justifying steps (again)

Comment

Things that were previously taken as obvious, for which there was no proof (because this had not yet been found) would be used in a solution without stating why they were being used.

When those things became proved, they became theorems. Any subsequent step in a solution which used these theorems then had to be *justified by stating the relevant theorem* being used.

Conventional use of symbols

Example : Assume that there is a question for which the first step of the solution is the following:

Solution : Let $b(\ell) = i(\ell).o(\ell) + \sigma(\ell)$

How could the presentation of this step be improved?

Conventional use of symbols

The presentation could be improved this way:

Solution: let $f(x) = g(x)\,Q(x) + R(x)$

So, for the sake of ease of readability it is best to use conventional symbols.

Conventional use of symbols

Exercise: How could the presentation of the step be improved?

Solution : $f(j) = (j-\alpha)\, Q(j) + R(j)$

Question: How is *j* being used? Is it a constant or does it represent the independent variable?

Conventional use of symbols

- Although technically correct, we don't write maths with non-conventional symbols. One of the reasons is to do with the making mathematics readable via the notation that is used.

- Also, in cswk/exam questions it is best to use the notation that has been used in the question (and not to alter the notation, even if this altered notation is also conventional).

Conventional use of symbols

The following two function statements are appropriate in terms of presentation/notation:

Solution: Let $f(x) = \sin x$

Let $f(z) = \sin z$

Conventional use of symbols

The following two function statements are appropriate in terms of presentation/notation:

Solution: let $f(x) = \sin x$ where $x \in \mathbb{R}$

let $f(z) = \sin z$ where $z \in \mathbb{C}$

Conventional use of symbols

Convention: Amongst other common symbols:

- Letters “x” and “y” are commonly used for the variables of functions of a real variable;
- Letters “z” and “w” are commonly used for the variables of functions of a complex variable;
- Letters “f”, “g”, “h” are commonly used for functions of a real variable.
- Letters “u” and “v” are commonly used for the functions of a complex variable;

Examples and exercises

Example 1

Consider the following question

By using the rational root test factorise

$$f(x) = 2x^2 + 5x - 3 = 0.$$

Let us now study the solution handed out to identify the missing elements which make this solution incomplete.

Examples and exercises

Exercise 1

Consider the following question

> Factorise and find the roots of
>
> $$f(x) = 6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0,$$
>
> demonstrating the use of the rational root test at least once in your solution.

Find and correct the presentation errors in the solution handed out.

Examples and exercises

Exercise 2

Write a complete solution to the following problem:

Factorise and find the roots of

$$f(x) = x^3 + 3x^2 + 2x = 0.$$

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Appendix

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Summary

- We have previously focused on the following aspects of the presentation of maths solutions:
  - Writing in complete sentences;
  - Not using arrows in your solution (unless …?);
  - Justifying certain steps;
  - Using exact values where possible;
  - Correct use of symbols;
  - Making your presentation clean and readable;
  - Not writing in columns.

Justifying steps (again)

Example: "Find the x-value of the minimum of $f(x) = 3x^2 - 3x + 1 = 0$"

Solution:

Since $f(x) = 3x^2 - 3x + 1 = 0$,

$$\frac{df}{dx} = 6x - 3$$

For max / min / point of inflection we have $\frac{df}{dx} = 0$

$$\therefore \quad 6x - 3 = 0 \Rightarrow x = \frac{1}{2}$$

Justifying steps (again)

Example: "Find the x-value of the minimum of $f(x) = 3x^2 - 3x + 1 = 0$"

Missing a justifying step somewhere

Solution

For $f(x) = ax^2 + bx + c = 0$

max / min occur at $x = -\frac{b}{2a}$

$\therefore$ $x = \frac{1}{2}$ is the min point

Justifying steps (again)

Example: "Find the *x*-value of the minimum of $f(x) = 3x^2 - 3x + 1 = 0$"

Solution:

FOR $f(x) = ax^2 + bx + c = 0$

max / min occur at $x = -\frac{b}{2a}$

This → Hence $x = -\frac{-3}{6} = \frac{1}{2}$

Justifying steps (again)

Example: "Find the x-value of the minimum of $f(x) = 3x^2 - 3x + 1 = 0$"

Or:

FOR $f(x) = ax^2 + bx + c = 0$

max / min occur at $x = -\frac{b}{2a}$.

Or this →

Since $a = 3$ and $b = -3$, $x = \frac{1}{2}$.

Conventional vs arbitrary symbols

Compare the following two expressions:

Solution: let $f(n) = (n - \alpha) Q(n) + R(n)$

Solution: let $f(\alpha) = (\alpha - \alpha) Q(\alpha) + R(\alpha)$

Which seems mathematically more clear? Why?

Density of mathematical English and mathematical expressions

Example

How to start a mathematical sentence.

- Start each sentence with a word not a symbol.

- **Example**
  - "dy/dx is the derivative of f". **No**
  - "The derivative of f is dy/dx". **Yes**

  or
  - "Function f has derivative dy/dx". **Yes**

If – Then

If you use "if", then use "then".

- **Example**
  - "If x is odd, x^2 is odd". **No**
  - "If x is odd, then x^2 is odd". **Yes**

If – Then

- The “then” in the previous example may have been obvious so why use it? To see why, consider the following example:

  “If $a > 0$, $b > 0$, $a + b > 0$”.

  What does this mean?

  1) “If $a > 0$, then $b > 0$ and $a + b > 0$?

  or

  2) “If $a > 0$ and $b > 0$ then $a + b > 0$?